Axial Laminar Flow in an Eccentric Annulus: an Approximate Solution

ISMAIL TOSUN

Department of Chemical Engineering
Middle East Technical University
Ankara, Turkey

INTRODUCTION

While the problem of fluid flow in a concentric annulus has received much attention in the past, only in the last two decades an interest has started to develop for the same problem in an eccentric annulus. Redberger and Charles (1962), using a finite-difference method, calculated the volumetric flow rate in an eccentric annulus and showed that the displacement of the inner pipe from a concentric position increased the volumetric flow rate for a given pressure gradient. Therefore, flow through an eccentric annulus is of considerable importance as far as the energy economy is concerned.

The exact analysis of fluid flow in an eccentric annulus is rather complex. Iyoho and Azar (1981) represented an approximate slot flow model for the flow of non-Newtonian fluids through an eccentric annulus and determined the velocity profile. However, they calculated the volumetric flow rate using the modified version of the formula given by Skelland (1972) in which the eccentricity ratio was not taken into account.

In the present analysis, the method of Iyoho and Azar (1981) is extended to calculate the approximate volumetric flow rate through an eccentric annulus as functions of eccentricity ratio and radius ratio. The results are shown to be in agreement with the exact values. An exact analytical expression for the volumetric flow rate is also presented.

CALCULATION OF THE VOLUMETRIC FLOW RATE

Case I: Exact Solution

Snyder and Goldstein (1965), using the bipolar coordinate system, determined the velocity distribution for the fully-developed laminar flow of a Newtonian fluid in an eccentric annulus as

$$v_z = \left(-\frac{dP}{dz}\right)\frac{a^2}{\mu}\left\{A\eta + B - \frac{\coth\eta}{2} + \sum_{n=1}^{\infty}\left[C_n e^{n\eta} + (D_n - \coth\eta)e^{-n\eta}\right]\cos n\xi\right\} \quad (1)$$

where

$$A = \frac{\coth\eta_i - \coth\eta_o}{2(\eta_i - \eta_o)} \quad (2)$$

$$B = \frac{\eta_i \coth\eta_o - \eta_o \coth\eta_i}{2(\eta_i - \eta_o)} \quad (3)$$

$$C_n = \frac{\coth\eta_i - \coth\eta_o}{e^{2n\eta_i} - e^{2n\eta_o}} \quad (4)$$

$$D_n = \frac{\coth\eta_o e^{2n\eta_i} - \coth\eta_i e^{2n\eta_o}}{e^{2n\eta_i} - e^{2n\eta_o}} \quad (5)$$

The volumetric flow rate can be determined from the expression

$$Q = a^2 \int_{\eta_o}^{\eta_i}\int_0^{2\pi} \frac{v_z}{(\cosh\eta - \cos\xi)^2}\, d\xi\, d\eta \quad (6)$$

The analytical expression for the volumetric flow rate is rather complex and was not presented by Snyder and Goldstein (1965). After a lengthy treatment it can be shown that the volumetric flow rate is

$$Q = \left(-\frac{dP}{dz}\right)\frac{\pi a^4}{2\mu}\phi \quad (7)$$

where

$$\phi = (\coth\eta_i - \coth\eta_o)\left[\frac{1}{\eta_o - \eta_i} - 2\sum_{n=1}^{\infty}\frac{2n}{e^{2n\eta_i} - e^{2n\eta_o}}\right] + \frac{1}{4}\left(\frac{1}{\sinh^4\eta_o} - \frac{1}{\sinh^4\eta_i}\right) \quad (8)$$

On the other hand, the volumetric flow rate in a concentric annulus is given by (Bird, Stewart and Lightfoot, 1960)

$$Q = \left(-\frac{dP}{dz}\right)\frac{\pi r_o^4}{8\mu}\left[1 - r^{*4} + \frac{(1 - r^{*2})^2}{ln r^*}\right] \quad (9)$$

Hence, the ratio of the volumetric flow rates in eccentric and concentric annuli, for the given pressure gradient, becomes

$$\frac{Q_{ecc}}{Q_{conc}} = \frac{4\sinh^4\eta_o}{1 - r^{*4} + [(1 - r^{*2})^2/ln r^*]}\phi \quad (10)$$

Case II: Approximate Solution

When an annulus is very thin, it may to a good approximation be considered as a thin slit. Tao and Donovan (1955), following the same lines, considered an eccentric annulus as a slit of variable height, Figure 1. The slit height, h, is given by

$$h = (r_o - r_i)(1 + \epsilon\cos\theta) \quad (11)$$

and is valid only for small values of $(r_o - r_i)$ and ϵ. Iyoho and Azar (1981) developed a more general expression for h as

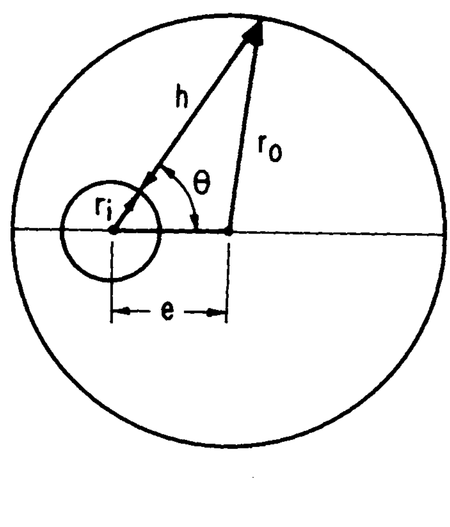
$$h = r_o\left[\sqrt{1 - k^2\sin^2\theta} + k\cos\theta - r^*\right] \quad (12)$$

where

$$k = \epsilon(1 - r^*) \quad (13)$$

Note that the term $\sqrt{1 - k^2\sin^2\theta}$ approaches unity for small values of k and Eq. 12 reduces to Eq. 11.

For the flow geometry shown in Figure 2, z component of the equation of motion



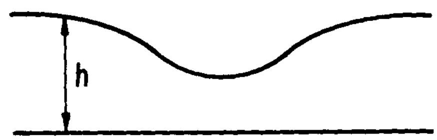


Figure 1. Slit equivalent of an eccentric annulus.

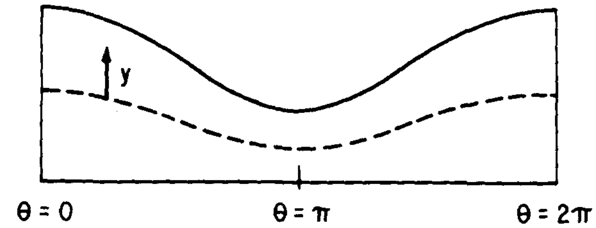


Figure 2. Flow geometry for a slit.

$$0 = -\frac{dP}{dz} + \mu \frac{d^2 v_z}{dy^2} \quad (14)$$

together with the boundary conditions

$$v_z = 0 \quad \text{at} \quad y = h/2 \quad (15)$$

$$dv_z/dy = 0 \quad \text{at} \quad y = 0 \quad (16)$$

has the solution of

$$v_z = \left(-\frac{dP}{dz}\right)\frac{1}{2\mu}\left[\left(\frac{h}{2}\right)^2 - y^2\right] \quad (17)$$

The flow area of the slit must be equal to the flow area of the eccentric annulus such that

$$\pi(r_o^2 - r_i^2) = \lambda \int_0^{2\pi} \int_0^h dy d\theta \quad (18)$$

where λ is the correction factor which is a function of ϵ and r^* From Eq. 18 λ becomes

$$\lambda = \frac{\pi r_o}{2} \frac{1 - r^{*2}}{(2E - \pi r^*)} \quad (19)$$

where

$$E = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\theta}\, d\theta \quad (20)$$

is the complete elliptic integral of the second kind.

The volumetric flow rate can be calculated from the formula

$$Q = 2\lambda \int_0^{2\pi} \int_0^{h/2} v_z dy d\theta \quad (21)$$

The use of Eq. 17 in Eq. 21 results in

$$Q = \left(-\frac{dP}{dz}\right) \frac{\pi r_o^4 (1 - r^{*2})}{12\mu(2E - \pi r^*)} H \quad (22)$$

where

$$H = 2E\left(\frac{k^2 + 7}{3} + 3r^{*2}\right) - \frac{8}{3}(1 - k^2)K - \pi r^*(3 + r^{*2}) \quad (23)$$

The term K in Eq. 23 is the complete elliptic integral of the first kind defined by

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} \quad (24)$$

For this case the ratio Q_{ecc}/Q_{conc} becomes

$$\frac{Q_{ecc}}{Q_{conc}} = \frac{2(1 - r^{*2})}{3(2E - \pi r^*)} \frac{H}{1 - r^{*4} + [(1 - r^{*2})^2/ln r^*]} \quad (25)$$

The ratio of the volumetric flow rates in eccentric and concentric annuli using Eqs. 10 and 25 is shown in Figure 3 as a function of r^* and ϵ. The exact and the approximate results are almost equal to each other for $r^* \geqslant 0.3$.

NOTATION

a = dimensionless x coordinate of pole at $\eta = \infty$
e = eccentricity
E = complete elliptic integral of the second kind, defined by Eq. 20
h = slit height
H = function defined by Eq. 23
k = function defined by Eq. 13
K = complete elliptical integral of the first kind, defined by Eq. 24

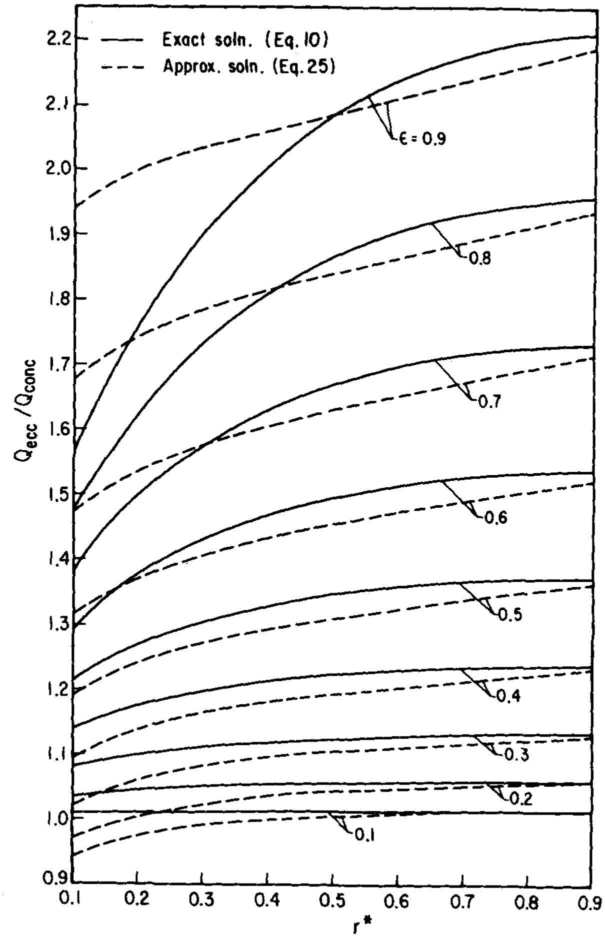


Figure 3. The ratio of the volumetric flow rates in eccentric and concentric annuli in terms of the diameter ratio and the eccentricity ratio.

P = pressure
Q = volumetric flow rate
r_i = radius of inner pipe
r_o = radius of outer pipe
r^* = radius ratio, r_i/r_o
v_z = z component of velocity
x,y,z = rectangular coordinates

Greek Letters

ϵ = eccentricity ratio, $e/(r_o - r_i)$
η,ξ = bipolar coordinates
θ = angle
λ = function defined by Eq. 19
μ = viscosity
ϕ = function defined by Eq. 8

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